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Coarser Connected Metrizable Topologies

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Abstract

We show that every metric space, X, with $w(X) \geq \mathfrak{c}$ has a coarser connected metrizable topology.

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1. Introduction, Definitions, Notation

The study of coarser connected topologies was started by Tkačenko, Tkachuk, and Uspenskij who developed necessary and sufficient conditions for a topological space to have coarser connected Hausdorff or regular topology [5]. Continuing the development, Gruenhage, Tkachuk, and Wilson showed that any noncompact metrizable space has a coarser connected Hausdorff topology [4]. Fleissner, Porter and Roitman showed that any zero-dimensional metrizable space with weight at least $\mathfrak{c}$ has a coarser connected metrizable topology [3]. We improve the last result by removing the zero-dimensional hypothesis. We show that every metric space with weight at least $\mathfrak{c}$ has a coarser connected metrizable topology.

We write (X, τ, μ) for a metric space X with metric μ and corresponding topology τ. For a set A and cardinal λ, we write $[A]^{\lambda}$ for the collection of subsets of A of cardinality λ and $[A]^{<\lambda}$ for the collection on subsets of cardinality less than λ. The extent, $e(X)$, of a space X is the supremum of cardinals, λ, such that there is a closed discrete subset of X of size λ. If a space, X, has a closed discrete set of size $e(X)$ we say the extent of X is attained in X.

Suppose U is a subset of a space X. If C is a closed discrete (with respect to U) subset of U, then C is not necessarily closed in X. Hence, it is possible that $e(U)$ is attained in U, but there is no closed discrete (in X) subset of U of

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size $e(U)$. If, on the other hand, there is a closed discrete (in X) subset of U of size $e(U)$, we say that $e(U)$ is attained in X.

For a cardinal κ and positive real number ϵ, the hedgehog with spininess κ and spine length ϵ is the space $Z = \{(\alpha, r) : \alpha \in \kappa, r \in (0, \epsilon]\} \cup \{(0,0)\}$ with metric ρ defined as follows:

$$\rho((\alpha, r), (\alpha', r')) = \begin{cases} r + r' & \text{if} \quad \alpha \neq \alpha' \\ |r - r'| & \text{if} \quad \alpha = \alpha'. \end{cases}$$

The hedgehog space is a connected metric space.

2. Some Lemmas

If X is a metric space such that $w(X) < \mathfrak{c}$, then X does not necessarily have a coarser connected metrizable topology. For example the disjoint union of countably many Cantor sets is a separable, metrizable, disconnected space with no coarser connected regular topology [5]. We aim to show that all metric spaces, X, with $w(X) \geq \mathfrak{c}$ have a coarser connected metrizable topology. This result has already been proven if $e(X)$ is attained (Druzhinina [1], Fleissner, Porter, Roitman [3]). If $e(X) = \mathfrak{c}$, then we will see later that $e(X)$ is attained. So, we consider metric spaces, X, such that $e(X) > \mathfrak{c}$ is not attained.

In the proof of the main theorem, we partition the space (X, τ, μ) into subsets W_i for $i \in \omega$ such that $e(W_i)$ is attained in X. Lemma 2.1 will be used to define a coarser topology on X in which W_i is connected. We show that μ can be adjusted so that W_i is connected without changing μ on $X \setminus W_i$.

Lemmas 2.2–2.5 discuss the consequences of $e(X)$ not being attained and provide a tool for the partition of X into $W_i's$. We apply Lemma 2.6 to a particular compact subset of X for the purpose of defining the partition of X into $W_i's$.

Lemma 2.1. *Suppose that Y is a subset of a metric space (X, τ, μ), $e(\mathrm{cl}Y) \geq \mathfrak{c}$ is attained in X by $C \subset \mathrm{int}(Y)$ and $\mathrm{diam}_\mu(\mathrm{cl}Y) \leq \epsilon$. If $U \subset Y$ is an open set containing C, then there is a coarser topology τ' on X generated by a corresponding metric μ' such that*

i) $\mu' \leq \mu + 2\epsilon$,

ii) $\mathrm{cl}_{\tau'}Y = \mathrm{cl}_\tau Y$ *and* $\tau'\big|_{\mathrm{clY}}$ *is connected, and*

iii) $\mu\big|_{(X\setminus U)^2} = \mu'\big|_{(X\setminus U)^2}$.

Proof. Let $e(\mathrm{cl}Y) = \kappa$. So, C is closed discrete in X and $|C| = \kappa$. By replacing C with a subset, we may assume that $|Y \setminus C| \geq \kappa$. Let $\mathcal{U} = \{U_c \subset U : c \in C\}$ be a discrete in X collection of open sets in X such that $c \in$

$U_c \subset \mathrm{cl}(U_c) \subset U$. For each $c \in C$ define a continuous function $f_c : X \to [0, \epsilon]$ such that $f_c(c) = \epsilon$ and $f_c[X \setminus U_c] = \{0\}$. For each $x, y \in X$ define $\mu^*(x, y) = \Sigma_{c \in C}|f_c(x) - f_c(y)| + \mu(x, y)$. Because C is closed discrete in X, it is easy to check that μ^* generates τ, $\mu^*\big|_{(X \setminus U)^2} = \mu\big|_{(X\setminus U)^2}$ and $\mu^* \leq \mu + 2\epsilon$.

Let (Z, ρ) be a hedgehog space with spininess κ and spine length ϵ. Let $T = \{(\alpha, \epsilon) : \alpha \in \kappa\}$ and let $S = Z \setminus T$. The set T is the set of tips of the spines of the hedgehog which is a closed discrete set of size κ. Let $D \subset Y \setminus C$ be a dense subset of $Y \setminus C$ of size $d(Y) = e(Y) = \kappa$. Define a one-to-one map $f : Z \to Y$ such that $f[T] = D$ and $f[S] = C$. For x, y in the image if f, let $\lambda(x, y) = \min\{\mu^*(x, y), \rho(f^{-1}(x), f^{-1}(y))\}$. For all other $x, y \in X$, let $\lambda(x, y) = \mu^*(x, y)$.

Define a function μ' on $X \times X$ as follows:

$$\mu'(x, y) = \inf\{\lambda(x, x_1) + \lambda(x_1, x_2) + \cdots + \lambda(x_{n-1}, x_n) + \lambda(x_n, y)\}$$

where $x_1, x_2, \ldots, x_n$ ranges over all finite sequences (including the empty sequence) of distinct elements of $X \setminus \{x, y\}$. Since ρ and μ^* satisfy the triangle inequality, in defining μ' it suffices to consider sums of the form

$$\begin{aligned} \mu^*(x, x_1) + \rho(f^{-1}(x_1), f^{-1}(x_2)) + \cdots + \mu^*(x_{n-1}, x_n) \\ + \rho(f^{-1}(x_n), f^{-1}(y)) \end{aligned} \tag{1}$$

where the sum may start or end with either a ρ term or μ^* term and the terms of the sum alternate between ρ and μ^*. Since $x_1, x_2, \ldots, x_n$ are in the image of f, note that $x_1, x_2, \ldots, x_n \in D \cup C$. Moreover, if $x \notin Y$ and $y \in Y$ each sum begins with a μ^* term and $x_1 \in Y$. Hence, $\mu'(x, y) \geq \mu^*(x, Y) \geq \mu(x, Y)$. Letting $y \in Y$ vary, we have $\mu'(x, Y) \geq \mu(x, Y)$.

Now we verify that μ' is a metric. Since μ' is an infimum over all finite sequences in X, it satisfies the triangle inequality. It is clear that $\mu'(x, y) = \mu'(y, x)$ and $\mu'(x, x) = 0$ for all $x, y \in X$. It remains to show that $\mu'(x, y) = 0$ implies that $x = y$.

Claim 1. *Suppose that for a particular sum in the form of (1) and some $m < n$, $\lambda(x_i, x_{i+1}) < \epsilon$ for each $i \in \{m, m+1, \ldots, n-1\}$. Then, if $\lambda(x_m, x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$ and $x_m \in D$, then $x_{m+i} \in C$ for all i odd and $x_{m+i} \in D$ for all i even in $\{0, 1, \ldots, n-m\}$.*

Proof. Suppose that $\lambda(x_m, x_{m+1}) = \rho(f^{-1}(x_m), f^{-1}(x_{m+1}))$ and $x_m \in D$. Also suppose that $\lambda(x_i, x_{i+1}) < \epsilon$ for each $i \in \{m, m+1, \ldots, n-1\}$. Fix i even such that $0 \leq i < n-m$, and suppose that $x_{m+i} \in D$. Since $x_{m+i} \in D$, $f^{-1}(x_{m+i}) \in T$. But since the ρ and μ terms of the sum alternate,

$$\lambda(x_{m+i}, x_{m+i+1}) = \rho(f^{-1}(x_{m+i}), f^{-1}(x_{m+i+1})) < \epsilon$$

and hence $f^{-1}(x_{m+i+1}) \in S$. By the definition of f, $x_{m+i+1} \in C$. Now, since $\lambda(x_{m+i+1}, x_{m+i+2}) = \mu^*(x_{m+i+1}, x_{m+i+2}) < \epsilon$ and $x_{m+i+1} \in C$, it must be that $x_{m+i+2} \in D$. So, since $x_m \in D$, by induction, $x_{m+i} \in C$ for all i odd and $x_{m+i} \in D$ for all i even in $\{0, 1, \ldots, n-m\}$. $\square$

Claim 2. $\mu'(x, y) = 0$ *implies that* $x = y$.

Proof. Suppose there were $x, y \in X$ such that $\mu'(x, y) = 0$ but $x \neq y$. If defined, let $x' = f^{-1}(x)$ and $y' = f^{-1}(y)$. Let $x_1, \ldots x_n \in D \cup C$ be a sequence that yields an alternating μ^*, ρ sum between x and y that is less than

$$\delta(x, y) = \min\{\epsilon, \mu^*(x, C\backslash\{x\}), \mu^*(y, C\backslash\{y\}), \rho(x', T\backslash\{x'\}), \rho(y', T\backslash\{y'\})\}.$$

Since μ^* and ρ are metrics, we see that $n \geq 1$. Since C is closed discrete in X and T is closed in Z, $\delta(x, y)$ is a positive real number.

Case 1. The alternating sum begins and ends with μ^* terms.

If $x_1 \in C$ then $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta(x, y)$, which is a contradiction.

Note that $n \geq 2$ is even. If $x_1 \in D$, then since $\lambda(x_i, x_{i+1}) < \epsilon$ for each $1 \leq i < n$ and $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$, by Claim 1, $x_n \in C$. Hence $\mu^*(x_n, y) \geq \mu^*(y, C \setminus \{y\}) \geq \delta(x, y)$, which is a contradiction.

Case 2. The sum begins with a μ^* term and ends with a ρ term.

If $x_1 \in C$ then $\mu^*(x, x_1) \geq \mu^*(x, C \setminus \{x\}) \geq \delta(x, y)$, which is a contradiction.

Note that n is odd. If $x_1 \in D$, then since $\lambda(x_i, x_{i+1}) < \epsilon$ for each $1 \leq i < n$ and $\lambda(x_1, x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$, by Claim 1, $x_n \in D$. Then $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$, which is a contradiction.

A symmetric argument shows that the sum beginning with ρ and ending with μ^* yields a contradiction.

Case 3. The sum begins and ends with ρ terms.

Suppose $x_1 \in D$. Then $\rho(f^{-1}(x), f^{-1}(x_1)) = \rho(x', f^{-1}(x_1)) \geq \rho(x', T \setminus \{x'\}) \geq \delta(x, y)$, which is a contradiction.

Suppose $x_1 \in C$. Since $\lambda(x_1, x_2) = \mu^*(x_1, x_2) < \epsilon$, $x_2 \in D$. Note that $n \geq 2$ is even. If $n = 2$, then since $x_2 \in D$, $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$, which is a contradiction. If $n > 2$, $\lambda(x_i, x_{i+1}) < \epsilon$ for each i with $2 \leq i < n$ and $\lambda(x_2, x_3) = \rho(f^{-1}(x_2), f^{-1}(x_3))$. Therefore, $x_n \in D$ by Claim 1. Hence $\rho(f^{-1}(x_n), y') \geq \rho(y', T \setminus \{y'\}) \geq \delta(x, y)$, which is a contradiction. $\square$

So, μ' defines a metric on X. Let τ' be the topology on X generated by μ'. Since μ^* generates τ and $\mu' \leq \mu^*$, τ' is coarser than τ. Since $\mu' \leq \mu^*$

and $\mu^* \leq \mu + 2\epsilon$, we have that $\mu' \leq \mu + 2\epsilon$ showing i) of the lemma.

Note 1. In the proof of Claim 2, we have actually shown that for all $x, y \in X$

$$\min\{\mu^*(x,y), \delta(x,y)\} \leq \mu'(x,y) \leq \mu^*(x,y).$$

Claim 3. *The map* $f : (Z, \rho) \to (Y, \mu'|_Y)$ *is continuous.*

Proof. Let $x \in Y$, $\delta > 0$ and let $U = B_{\mu'}(x, \delta)$ be the μ'-ball of radius δ about x. Suppose $z \in f^{\leftarrow}[U]$. Then $\mu'(f(z), x) = \delta' < \delta$. Let $\xi = (\delta - \delta')/2$. Suppose $z' \in Z$ satisfies $\rho(z, z') < \xi$. We wish to show that $\mu'(f(z'), x) < \delta$ so that $B_\rho(z, \xi) \subset f^{\leftarrow}[U]$. Since $\mu'(x, f(z)) = \delta'$, there is a sequence $x_1, \dots, x_n \in C \cup D \subset Y$ such that

$$\lambda(x, x_1) + \cdots + \lambda(x_n, f(z)) < \delta' + \xi.$$

Adding the term $\rho(z, z') = \rho(f^{-1}(f(z)), f^{-1}(f(z')))$ to this sum,

$$\lambda(x, x_1) + \cdots + \lambda(x_n, f(z)) + \rho(z, z') < \delta' + 2\xi < \delta$$

illustrates that the sequence $x_1, \dots, x_n, f(z)$ yeilds a sum between x and $f(z')$ that is less than δ. Hence, $\mu'(x, f(z')) < \delta$ as desired. So, the map f is continuous and the claim is proven. $\square$

Since (Z, ρ) is connected, $\tau \subset \tau'$ and $C \cup D$ is dense in $(Y, \tau|_Y)$, by Claim 3, $(Y, \mu'|_Y)$ is connected. For $x \notin Y$, we argued before that $\mu'(x, Y) \geq \mu(x, Y)$, so $\mathrm{cl}_{\tau'} Y \subset \mathrm{cl}_\tau Y$. However $\mathrm{cl}_\tau Y \subset \mathrm{cl}_{\tau'} Y$, since τ' is coarser that τ. Hence $\mathrm{cl}_{\tau'} Y = \mathrm{cl}_\tau Y = \mathrm{cl}_\tau(D \cup C)$ and therefore $(\mathrm{cl}Y, \tau'\big|_{\mathrm{cl}Y})$ is connected. Therefore, ii) of the lemma is shown.

We now show iii). Let $x, y \notin U$. We verify that $\mu'(x, y) = \mu(x, y)$. By definition, $\mu'(x, y) \leq \mu^*(x, y) = \mu(x, y)$ so we need only show that $\mu'(x, y) \geq \mu^*(x, y)$.

Suppose for a contradiction that $\mu'(x, y) < \mu^*(x, y)$. In other words, there exist $x_1, \dots, x_n \in D \cup C$ such that

$$\mu'(x, y) \leq \lambda(x, x_1) + \cdots + \lambda(x_n, y) < \mu^*(x, y).$$

Since $x, y \notin U$, $\mu^*(x, y) = \mu(x, y)$. If $f^{-1}(x)$ and $f^{-1}(y)$ are defined, then since $x, y \notin C$, $\rho(f^{-1}(x), f^{-1}(y)) = 2\epsilon > \mu(x, y) = \mu^*(x, y)$. Hence, the sequence $x_1, \dots, x_n$ is not empty. Now, since $\mathrm{diam}_\mu(Y) \leq \epsilon$, for any $z_1, z_2 \in Y$,

$$\mu(x, y) \leq \mu(x, z_1) + \epsilon + \mu(z_2, y).$$

Hence,

$$\mu(x, y) \leq \mu(x, Y) + \epsilon + \mu(y, Y).$$

Combining these inequalities we have

$$\mu'(x,y) \leq \lambda(x,x_1) + \cdots + \lambda(x_n,y) < \mu(x,Y) + \epsilon + \mu(y,Y). \tag{2}$$

Case 1. $x, y \in Y$.

Then inequality (2) becomes $\mu'(x,y) \leq \lambda(x,x_1) + \cdots + \lambda(x_n,y) < \epsilon$. Since $x, y \notin U$, $\mu^*(x, C \setminus \{x\}) = \mu^*(x,C) \geq \epsilon$ and $\mu^*(y, C \setminus \{y\}) = \mu^*(y,C) \geq \epsilon$. Since $x \notin C$, if $x' = f^{-1}(x)$ is defined, then $x \in D$. Hence $\rho(x', T \setminus \{x'\}) \geq \epsilon$. Similarly, if $y' = f^{-1}(y)$ is defined, $\rho(y', T \setminus \{y'\}) \geq \epsilon$. So, $\delta(x,y) =$
$\min\{\epsilon, \mu^*(x, C \setminus \{x\}), \mu^*(y, C \setminus \{y\}), \rho(x', T \setminus \{x'\}), \rho(y', T \setminus \{y'\})\} \geq \epsilon$. By Note 1 after Claim 2, $\mu'(x,y) \geq \min\{\mu^*(x,y), \epsilon\}$. This contradicts the assumption that $\mu'(x,y) < \mu^*(x,y)$.

Case 2. $x \notin Y$ and $y \in Y$.

Since $y \in Y$, $\mu(y,Y) = 0$, hence (2) becomes $\mu'(x,y) \leq \lambda(x,x_1) + \lambda(x_1,x_2) + \cdots + \lambda(x_n,y) < \mu(x,Y) + \epsilon$. Suppose $x_1 \in C$. Since $x \notin Y$, $\lambda(x,x_1) = \mu^*(x,x_1) = \mu(x,x_1) + \epsilon \geq \mu(x,Y) + \epsilon$, a contradiction. Suppose that $x_1 \in D$. Since $\lambda(x,x_1) = \mu^*(x,x_1) \geq \mu(x,Y)$ we have that $\lambda(x_1,x_2) + \cdots + \lambda(x_n,y) < \epsilon$. If n is even, $\lambda(x_n,y) = \mu^*(x_n,y)$ and by Claim 1, $x_n \in C$. Therefore $\lambda(x_n,y) = \mu^*(x_n,y) = \mu(x_n,y) + \epsilon > \epsilon$, a contradiction. If n is odd, $\lambda(x_n,y) = \rho(f^{-1}(x_n), f^{-1}(y))$ and so $y \in C \cup D$. Since $y \notin U$, we have that $y \in D$ and therefore $\rho(f^{-1}(x_n), f^{-1}(y)) = 2\epsilon > \epsilon$, a contradiction.

A symmetric argument yields a contradiction in the case that $x \in Y$ and $y \notin Y$.

Case 3. $x, y \notin Y$.

Since $x, y \notin Y$, they are not in the image of f and therefore the alternating sum must start and end with μ^* terms implying n is even and $n \geq 2$. Since $x_1, x_n \in C \cup D \subset Y$, $\mu^*(x,x_1) \geq \mu(x,x_1) \geq \mu(x,Y)$ and $\mu^*(x_n,y) \geq \mu(x_n,y) \geq \mu(y,Y)$. Combining this with (2) gives:

$$\rho(f^{-1}(x_1), f^{-1}(x_2)) + \mu^*(x_2,x_3) + \cdots + \rho(f^{-1}(x_{n-1}), f^{-1}(x_n)) < \epsilon,$$

$$\mu^*(x,x_1) < \mu(x,Y) + \epsilon \tag{3}$$

and

$$\mu^*(x_n,y) < \mu(y,Y) + \epsilon. \tag{4}$$

Suppose $x_1 \in C$. Then $\mu^*(x,x_1) = \mu(x,x_1) + \epsilon \geq \mu(x,Y) + \epsilon$ contradicting (3). So, $x_1 \notin C$. Similarly, $x_n \in C$ contradicts (4). So, $x_n \notin C$. Hence we have shown that $x_1, x_n \in D$. Now, $\lambda(x_i, x_{i+1}) < \epsilon$ for each i with $1 \leq i < n$ and $\lambda(x_1,x_2) = \rho(f^{-1}(x_1), f^{-1}(x_2))$. So, since $x_1 \in D$ and n is even, by Claim 1, $x_n \in C$, which is a contradiction. Therefore, $\mu'(x,y) = \mu^*(x,y) = \mu(x,y)$, and iii) of the lemma follows. $\square$

If X is a metric space with $e(X)$ not attained, then X must have a certain form. This form is described in the next lemma.

Lemma 2.2. (Fitzpatrick, Gruenhage, Ott [2]) *Let (X, τ) be a metric space with metric μ in which $e(X) = \kappa$ is not attained. Let K be the set of points x of X such that every neighborhood of x has extent κ. Then*
(1) κ is a singular cardinal of cofinality ω.
(2) K is a compact, nowhere dense subset of X.
(3) If U is an open subset of X such that $\mathrm{cl}_\tau U \cap K = \emptyset$, then $e(U) < \kappa$.
(4) K is nonempty.

Recall that Konig's Lemma implies that $\mathrm{cf}(\mathfrak{c}) > \omega$. So, if (X, τ) is a metric space and $e(X) = \mathfrak{c}$, by Lemma 2.2, the extent of X must be attained.

Definition 2.3. (see [3]) An open set V is called *e-homogeneous* if for every nonempty open subset V' of V, $e(V') = e(V)$.

Suppose U is a nonempty open subset of a metric space. We argue that there is a nonempty open e-homogeneous set V such that $V \subset U$. If U is e-homogeneous then set $V = U$. If U is not e-homogenous, let $V_0 = U$ and let V_1 be a nonempty open subset of U such that $e(V_1) < e(U)$. Continue by defining $V_n \subset V_{n-1}$ such that $e(V_n) < e(V_{n-1})$ if V_{n-1} is not homogeneous. Since $e(V_1), e(V_2), \ldots$ is a decreasing sequence of of cardinal numbers, there is $n \in \omega$ such that $V = V_n$ is e-homogeneous.

Suppose the extent of an open subset, U, of a metric space is not attained in U. Apply Lemma 2.2 with $X = U$. Then, by property (2), there is $x \in U \setminus K$ and an open set $V \subset U$ with $e(V) < e(U)$ and $x \in V$. Therefore, if U is an e-homogeneous subset of a metric space, $e(U)$ is attained in U. However, as mentioned before, this does not mean that $e(U)$ is attained in X.

Lemma 2.4. *Let U be an e-homogeneous subset of a metric space (X, τ, μ) with metric μ and $e(U) > \aleph_0$. Then $e(U)$ is attained in X.*

Proof. Since U is e-homogeneous, $e(U) = \lambda$ is attained in U. Suppose that $cf(\lambda) > \aleph_0$. Then, $e(U)$ is attained by some closed discrete (in U) subset $C' \subset U$ of cardinality λ. However, it may not be the case that C' is closed discrete in X. We will construct a set $C \subset U$ such that C is closed discrete in $\mathrm{cl}_\tau U$, hence closed discrete in X, and $|C| = e(U) = \lambda$. Let $\mathcal{U}$ be a discrete collection of open subsets of U that witnesses that C' is closed discrete in U. Let

$$L = \mathrm{cl}_\tau\left(\bigcup \mathcal{U}\right) \setminus \bigcup_{U \in \mathcal{U}} \mathrm{cl}_\tau U.$$

Note that $L \subset \mathrm{cl}_\tau U \setminus U$. If $L = \emptyset$ then $\mathcal{U}$ is discrete in X and C' is closed discrete in X. Therefore, suppose that $L \neq \emptyset$. Let $C_n = \{c \in C' : \mu(c, L) \geq$

$1/n\}$ and note that $\bigcup C_n = C'$. Since $cf(\lambda) > \aleph_0$ there is $n \in \omega$ so that $|C_n| = \lambda$. Set $C = C_n$. By construction, C is closed and $\mathrm{cl}_\tau C \cap L = \emptyset$, so C is closed discrete in X, $C \subset U$ and $|C| = \lambda = e(U)$ as desired.

Now suppose that $cf(\lambda) = \aleph_0$. Let $\{\lambda_n : n \in \omega\}$ be an increasing sequence of cardinals such that $\lambda = \sup_{n\in\omega} \lambda_n$. Let W be an open subset of U such that $\mathrm{cl}_\tau W \subset U$. Note that since U is e-homogeneous, $e(W) = e(\mathrm{cl}_\tau W) = e(U) = \lambda$. Since $e(\mathrm{cl}_\tau W) = \lambda > \aleph_0$ there is $A \subset \mathrm{cl}_\tau W \subset U$ a countable closed discrete set in $\mathrm{cl}_\tau W$, hence closed discrete in X. Let $\mathcal{U} = \{U_n : n \in \omega\}$ be a discrete collection of open subsets of U that witnesses A is discrete in X. For each $n \in \omega$, $e(\mathrm{cl}_\tau U_n) = \lambda > \lambda_n$, so there is $C_n \subset \mathrm{cl}_\tau U_n$ a closed discrete in X subset of cardinality λ_n. Let $C = \bigcup C_n$. Since C_n is closed discrete in $\mathrm{cl}_\tau U_n$, it is closed discrete in X. Moreover, since $\mathcal{U}$ is discrete, C is closed discrete in X. Finally, $|C| = \lambda = e(\mathrm{cl}_\tau U)$ by construction and since $U_n \subset U$ for each n, $C \subset U$. In either case, $|C| = \lambda = e(U)$, $C \subset U$ and C is closed and discrete in X. Hence, $e(U)$ is attained in X. □

Lemma 2.5. *Let (X, τ) be a metric space with metric μ in which $e(X) = \kappa$ is not attained. Let K be the set of points x of X such that every neighborhood of x has extent κ. Then, for every open set U meeting K and every $\theta < \kappa$ there is an open subset V of U such that $\mathrm{cl}_\tau V \subset U$, $e(V) > \theta$ is attained in X and $K \cap \mathrm{cl}_\tau V = \emptyset$.*

Proof. Let $\mathcal{V}$ be a maximal pairwise disjoint collection of e-homogeneous subsets V of $U \setminus K$ such that $\mathrm{cl}_\tau V \cap K = \emptyset$. For each $V \in \mathcal{V}$, since $\mathrm{cl}_\tau V \cap K = \emptyset$, by item (3) of Lemma 2.2, $e(V) < \kappa$. Suppose that for some $V \in \mathcal{V}$, $e(V) > \theta$. By Lemma 2.4, $e(V)$ is attained in X and we are done. Suppose on the other hand that $e(V') \leq \theta$ for all $V' \in \mathcal{V}$. Since $\mathcal{V}$ is maximal, K is nowhere dense and every open subset of X has an e-homogeneous subset, $\bigcup \mathcal{V}$ is dense in U. Since X is metric, $e(W) = d(W)$ for any open subset W. Suppose that $|\mathcal{V}| = \lambda < \kappa$. Then, $d(U) = d(\bigcup \mathcal{V}) \leq \lambda \cdot \theta < \kappa$ which is a contradiction. So, $|\mathcal{V}| = \kappa$. Since $\mu(V', K) > 0$ for all $V' \in \mathcal{V}$ and $cf(\kappa) = \omega$, there is $n \in \omega$ such that $|\{V' \in \mathcal{V} : \mu(K, V') > 1/n\}| > \theta$. Set $\mathcal{V}' = \{V' \in \mathcal{V} : \mu(K, V') > 1/n\}$. We now refine $\mathcal{V}'$ to a discrete collection of size $> \theta$. Let

$$L = \mathrm{cl}_\tau\big(\bigcup \mathcal{V}'\big) \setminus \bigcup_{V \in \mathcal{V}'} \mathrm{cl}_\tau V$$

and let $L_\epsilon = \{x \in U : \mu(x, L) \leq \epsilon\}$. Since $\bigcap\limits_{\epsilon > 0} L_\epsilon = L$ and $V' \cap L = \emptyset$ for all $V' \in \mathcal{V}'$, there is $m \in \omega$ such that $|\{V' \in \mathcal{V}' : V' \setminus L_{1/m} \neq \emptyset\}| > \theta$. Set $\mathcal{V}'' = \{V' \in \mathcal{V}' : V' \setminus L_{1/m} \neq \emptyset\}$. For each $V' \in \mathcal{V}''$, let $W(V') = V' \setminus L_{1/m}$. The collection $\mathcal{W} = \{W(V') : V' \in \mathcal{V}'\}$ by construction is discrete and has cardinality $> \theta$. Set $V = \bigcup \mathcal{W}$. For each $W \in \mathcal{W}$, choose $x_W \in W$. Let $C = \{x_W : W \in \mathcal{W}\}$. Then C is closed discrete in X since $\mathcal{W}$ is discrete in X, and $|C| = |\mathcal{W}| > \theta$. Since $e(V) \leq \sup\{e(W) : W \in \mathcal{W}\} \cdot |\mathcal{W}| \leq \theta \cdot |\mathcal{W}| = |C|$, we have that $e(V) > \theta$ is attained in X. □

Lemma 2.6. *Let K be a compact metric space with topology σ and let $\mathcal{U} \in [\sigma]^{<\omega}$ be a pairwise disjoint collection such that $\bigcup \mathcal{U}$ is dense in K. If $\epsilon > 0$ and $\mathcal{V}$ is the collection of open subsets of K with diameter less than ϵ, then there exists a pairwise disjoint $\mathcal{V}' \in [\mathcal{V}]^{<\omega}$ such that $\mathcal{V}'$ refines $\mathcal{U}$ and $\bigcup \mathcal{V}'$ is dense in K.*

Proof. Since $\mathcal{V}$ covers K, compact, there are $n \in \omega$ and $V_1, V_2, \ldots, V_n \in \mathcal{V}$ such that $K = \bigcup\{V_i : 1 \leq i \leq n\}$. Define $\hat{V}_1 = V_1$ and for $1 < i \leq n$ let $\hat{V}_i = V_i \setminus \mathrm{cl}(\bigcup\{V_j : 1 \leq j < i\})$. Let $\hat{\mathcal{V}} = \{\hat{V}_i : 1 \leq i \leq n\}$. Note that $\hat{\mathcal{V}}$ is pairwise disjoint and $\bigcup \hat{\mathcal{V}}$ is dense in X. Now define $\mathcal{V}' = \{V \cap U : V \in \hat{\mathcal{V}}, U \in \mathcal{U}\}$. Since $V \cap U \subset U$ for each $V \in \hat{\mathcal{V}}$ and $U \in \mathcal{U}$, $\mathcal{V}'$ refines $\mathcal{U}$. Since $\mathrm{diam}(V \cap U) \leq \mathrm{diam}(V) \leq V_i$ for some $1 \leq i \leq n$, $\mathrm{diam}(V') < \epsilon$ for all $V' \in \mathcal{V}'$. Hence $\mathcal{V}' \subset \mathcal{V}$. Since $\hat{\mathcal{V}}$ is pairwise disjoint, $\mathcal{V}'$ is as well. Finally, $\bigcup \mathcal{V}' = \bigcup \hat{\mathcal{V}} \cap \bigcup \mathcal{U}$ and therefore $\bigcup \mathcal{V}'$ is dense in K, since $\bigcup \mathcal{V}'$ and $\bigcup \mathcal{U}$ are open and dense in K. □

3. The Main Theorem

Theorem 3.1. *If (X, τ) is a metric space with metric μ and $w(X) = \kappa \geq \mathfrak{c}$, then there is σ, a topology on X coarser than τ, such that (X, σ) is connected and metrizable.*

Proof. As mentioned in the beginning of Section 2, the theorem has been proven in the case when $e(X)$ is attained. Therefore, we assume $e(X) > \mathfrak{c}$ is not attained. Re-scale μ so that $\mathrm{diam}_\mu(X) < 1/2$ by replacing it with $\frac{\mu}{2(1+\mu)}$. Let K be the set of points x of X such that every neighborhood of x has extent κ. By Lemma 2.2, K is compact.

Let $\mathcal{C}_0^* = \{K\}$. For each $n \in \omega \setminus \{0\}$, define $\mathcal{C}_n^* \subset \tau|_K$, a pairwise disjoint finite collection with the following properties.

- $\mathrm{cl}(\bigcup \mathcal{C}_n^*) = K$.
- $\mathcal{C}_n^*$ refines $\mathcal{C}_{n-1}^*$.
- $B \in \mathcal{C}_n^*$ implies $\mathrm{diam}(B) < 1/2^n$.

Let $n \in \omega \setminus \{0\}$. Apply Lemma 2.6 with $\sigma = \tau|_K$, $\mathcal{U} = \mathcal{C}_{n-1}^*$ and $\epsilon = 1/2^n$ to get $\mathcal{V}'$, a pairwise disjoint collection of open sets in K with diameter less that $1/2^n$ that refines $\mathcal{C}_{n-1}^*$ and whose union is dense in K. Set $\mathcal{C}_n^* = \mathcal{V}'$.

Definition of B_i's

For $n \in \omega$ enumerate the elements of $\mathcal{C}_n^*$ as $\mathcal{C}_n^* = \{B_i^* : i_n \le i < i_{n+1}\}$ with an increasing sequence of integers, i_n, where $i_0 = 0$. For each $i \in \omega$, let $L_i = \{x \in X : \mu(x, B_i^*) \le \mu(x, K \setminus B_i^*)\}$, the set of points in X that are closer to B_i^* than they are to its complement. Fix $n \in \omega$ and let $B_{i_n} = L_{i_n}$ and for each i with $i_n < i < i_{n+1}$, let $B_i = L_i \setminus \bigcup\{\mathrm{cl}(B_j) : i_n \le j < i\}$. Note that $B_0^* = K$, $B_0 = X$ and that $B_j^* \subset B_i^*$ implies $L_j \subset L_i$.

We define $\mathcal{C}_n = \{B_i : i_n \le i < i_{n+1}\}$ and verify the following.

(i) For each $n \in \omega$, $\mathcal{C}_n$ is pairwise disjoint.

(ii) For each $n \in \omega$, $\bigcup \mathcal{C}_n$ is dense in X.

(iii) For all $i \in \omega$, $\mathrm{int}(B_i) \cap K \ne \emptyset$.

From the definition of B_i and $\mathcal{C}_n$, (i) is clear. Towards (ii), let $n \in \omega$ and let $x \in X$. Since $\bigcup \mathcal{C}_n^*$ is dense in K, $\mu(x, K) = \mu(x, \bigcup \mathcal{C}_n^*)$. Since $\mathcal{C}_n^*$ is finite, there is i such that $i_n \le i < i_{n+1}$ and $\mu(x, K) = \mu(x, B_i^*)$. Therefore, $\mu(x, B_i^*) \le \mu(x, K \setminus B_i^*)$ and either $x \in B_i$ or $x \in \mathrm{cl}(\bigcup\{B_j : i_n \le j < i\})$. In either case, $x \in \mathrm{cl}(\bigcup \mathcal{C}_n)$. For (iii), note that $\mathrm{int}(B_i) \cap K = B_i^* \setminus \mathrm{cl}(\bigcup\{B_j^* : i_n \le j < i\}) = B_i^* \ne \emptyset$ since $\mathcal{C}_n^*$ is pairwise disjoint.

Let $U_i = \{x \in X : \mu(x, K) < 1/2^{i+1}\}$.

Claim 1. *For $n \in \omega,\ i_n \le i < i_{n+1},\ \mathrm{diam}_\mu(L_i \cap U_m) \le 3/2^n$ for any $m \ge n-1$.*

Proof. Note that if $x \in L_i$, $\mu(x, B_i^*) \le \mu(x, K \setminus B_i^*)$ which implies $\mu(x, K) = \mu(x, B_i^*)$. For i with $i_n \le i < i_{n+1}$, $m \ge n-1$, $x, y \in L_i \cap U_m$ and $\epsilon > 0$ there exist $x_0, y_0 \in B_i^*$ such that $\mu(x, x_0), \mu(y, y_0) < 1/2^{m+1} + \epsilon/2 \le 1/2^n + \epsilon/2$. So,

$$\mu(x, y) \le \mu(x, x_0) + \mu(y, y_0) + \mu(x_0, y_0) \le 2/2^n + \epsilon + 1/2^n \le 3/2^n + \epsilon.$$

Hence, $\mathrm{diam}_\mu(L_i \cap U_m) \le 3/2^n$. $\square$

Definition of W_i's

We define $\mathcal{W} = \{W_i : i \in \omega\}$, a pairwise disjoint collection of open subsets of X such that

(i) $\mathrm{cl}(W_i) \cap K = \emptyset$,

(ii) $e(W_i) > \mathfrak{c}$ is attained in X by $C_i \subsetneq W_i$,

(iii) for $i_n \le i < i_{n+1}$, $\mathrm{diam}_\mu(W_i) \le 3/2^n$ and

(iv) $\bigcup \mathcal{W}$ is dense in X.

Let $\hat{W}_0 = \emptyset$. Apply Lemma 2.5 with $\theta = \mathfrak{c}$ and $U = \text{int}(B_0) = X$ to get an open subset V of X such that $e(V) > \theta = \mathfrak{c}$ is attained in X by $C_0 \subset V$ and $\text{cl}_\tau V \cap K = \emptyset$. By replacing C_0 with a subset of the same cardinality, we may assume that $C_0 \subsetneq V$. Set $W_0 = V$ and $S_0 = V$. Let $k_0 = \min\{k \geq 1 : S_0 \subset X \setminus U_k\} \geq 1$. By definition W_0 is open, and since $\text{cl}V \cap K = \text{cl}(W_0) \cap K = \emptyset$, (i) holds. By the choice of W_0, (ii) holds and (iii) is trivial since $\text{diam}_\mu(X) \leq 1/2$.

Suppose we have defined W_i for all i with $0 \leq i < i_n$ so that (i), (ii), and (iii) are satisfied. Also suppose that $S_m = \bigcup\{W_i : i < i_{m+1}\}$ is dense in $X \setminus \text{cl}(U_{k_{m-1}+1})$ and that $k_m = \min\{k \geq m : S_m \subset X \setminus U_k\}$ whenever $0 < m < n$.

Let i be such that $i_n \leq i < i_{n+1}$. Let $\hat{W}_i = B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1}))$. Since $\text{int}(B_i) \cap K \neq \emptyset$ and $K \subset X \setminus \text{cl}(S_{n-1})$, $\text{int}(B_i) \setminus \text{cl}(S_{n-1})$ is an open set meeting K. So, apply Lemma 2.5 with $\theta = \max\{e(\hat{W}_i), \mathfrak{c}\}$ and $U = \text{int}(B_i) \setminus \text{cl}(S_{n-1})$ to get an open subset V of U such that $\text{cl}_\tau V \subset U$, $e(V) > \theta$ is attained in X by $C_i \subset V$ and $\text{cl}_\tau V \cap K = \emptyset$. As before, we may assume $C_i \subsetneq V$. Set $W_i = \hat{W}_i \cup V$. According to our choice, W_i satisfies (i) and (ii) for each i such that $i_n \leq i < i_{n+1}$. Set $S_n = \bigcup\{W_i : i_n \leq i < i_{n+1}\} \cup S_{n-1}$. Let $k_n = \min\{k \geq n : S_n \subset X \setminus U_k\}$. By Claim 1, $\text{diam}(B_i \cap U_{n-1}) \leq 3/2^n$ since $B_i \subset L_i$ and $i_n \leq i < i_{n+1}$. Also, S_{n-1} is dense in $X \setminus U_{k_{n-2}+1}$. Therefore, for any i as above, $W_i = \hat{W}_i \cup V \subset B_i \setminus \text{cl}(S_{n-1}) \subset B_i \cap U_{k_{n-2}+1} \subset B_i \cap U_{n-1}$, since $k_{n-2} \geq n-2$. Hence $\text{diam}_\mu(W_i) \leq 3/2^n$ and (iii) is satisfied.

Towards (iv), since K is nowhere dense in X, we only show that $\bigcup \mathcal{W}$ is dense in $X \setminus K$. Let $x \in X \setminus K$ and let $n \in \omega \setminus \{0\}$ be such that $x \in X \setminus \text{cl}(U_{k_{n-1}+1})$. Then either $x \in \text{cl}(S_{n-1}) \subset \text{cl}(S_n)$ or $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$. If $x \in X \setminus \text{cl}(U_{k_{n-1}+1} \cup S_{n-1})$ then since $\{B_i : i_n \leq i < i_{n+1}\}$ is dense in X, $x \in \text{cl}(B_i \setminus (\text{cl}(U_{k_{n-1}+1} \cup S_{n-1})) \subset \text{cl}(W_i) \subset \text{cl}(S_n)$ for some i. In either case $x \in \text{cl}(S_n)$. So,

$$x \in \text{cl}\big(\bigcup\{W_i : i \in \omega\}\big) = \text{cl}\big(\bigcup\{S_n : n \in \omega\}\big).$$

Hence $\bigcup\{W_i : i \in \omega\}$ is dense in $X \setminus K$ and in X.

Linking the W_i's

Suppose that $n, i, j \in \omega$ are such that $i_n \leq i < i_{n+1} \leq j < i_{n+2}$ and $B_j^* \subset B_i^*$. Then $W_i \subset B_i \cap U_{n-1} \subset L_i \cap U_{n-1}$ and $W_j \subset B_j \cap U_n \subset L_j \cap U_n \subset L_i \cap U_{n-1}$. Hence by Claim 1, $x \in W_i$ and $y \in W_j$ implies $\mu(x, y) \leq 3/2^n$. Since for each $j \in \omega$, $C_j \subsetneq W_j$, we may choose $x_j \in W_j \setminus C_j$ arbitrarily. For every i with $i_n \leq i < i_{n+1}$, let $J_i = \{j : i_{n+1} \leq j < i_{n+2}, B_j^* \subset B_i^*\}$ and let $X_i = \{x_j : j \in J_i\}$. Notice that $\text{diam}(W_i \cup X_i) \leq 3/2^n$, $C_i \subset W_i \subset \text{int}(W_i \cup X_i)$ and that by the definition of $\mathcal{B}^*$, $\bigcup_{i \in \omega} J_i = \omega \setminus \{0\}$.

Defining the connected topology on X

We define a sequence of metrics μ_n on X such that $\nu = \lim \mu_n$ is a well defined metric that generates a coarser connected topology on X. We define μ_n by induction.

Stage 0

Apply Lemma 2.1 with $Y = W_0 \cup X_0$, metric μ, topology τ, closed discrete set C_0, $\epsilon = 3/2$ and $U = W_0$ to get a topology τ_0 on X generated by the metric μ_0 on X such that

(i) $\mu_0 \leq \mu + 6/2$,

(ii) $\mathrm{cl}_\tau(W_0 \cup X_0) = \mathrm{cl}_{\tau_0}(W_0 \cup X_0)$ and $\tau_0\big|_{W_0 \cup X_0}$ is connected, and

(iii) $\mu\big|_{(X \setminus W_0)^2} = \mu_0\big|_{(X \setminus W_0)^2}$.

Note that since X_0 is finite, property (ii) implies that $\mathrm{cl}_{\tau_0}(W_0) = \mathrm{cl}_{\tau_0}(W_0 \cup X_0)$. So, for all $i \neq 0$, $\mathrm{int}_{\tau_0}(W_i) = W_i \setminus X_0 \supset W_i \setminus \{x_i\}$ and therefore since $x_i \notin C_i$, $C_i \subset \mathrm{int}_{\tau_0}(W_i)$.

Stage $n \geq 1$

Set $\sigma_0 = \tau_{n-1}$ and $\rho_0 = \mu_{n-1}$. For each i such that $i_n \leq i < i_{n+1}$, let $j = i - i_n$ and apply Lemma 2.1 with $Y = W_i \cup X_i$, $C = C_i$, $\tau = \sigma_j$, $\mu = \rho_j$, $U = \mathrm{int}_{\sigma_j}(W_i)$ and $\epsilon = 3/2^n$ to get a topology σ_{j+1} on X generated by the metric ρ_{j+1} on X such that

(i) $\rho_{j+1} \leq \rho_j + 6/2^n$,

(ii) $\mathrm{cl}_{\sigma_j}(W_i \cup X_i) = \mathrm{cl}_{\sigma_{j+1}}(W_i \cup X_i)$ and $\sigma_{j+1}\big|_{W_i \cup X_i}$ is connected, and

(iii) $\rho_j\big|_{(X \setminus \mathrm{int}_{\sigma_j}(W_i))^2} = \rho_{j+1}\big|_{(X \setminus \mathrm{int}_{\sigma_j}(W_i))^2}$.

Let $\mu_n = \rho_{m_n}$ and $\tau_n = \sigma_{m_n}$. Note that, as in Stage 0, property (ii) implies $C_i \subset \mathrm{int}_{\tau_n}(W_i)$ for all $i \geq i_{n+1}$. It is a consequence of (i) and (iii), that if $x \in X \setminus \bigcup\{W_i : i \leq i_{n+1}\}$ and $y \in \bigcup\{W_i : i < i_{n+1}\}$ then $\mu_n(x,y) \leq \mu(x,y) + 6/2^n$. Furthermore, $\mu_n = \mu_{n-1}$ on $X \setminus \bigcup\{W_i : i_n \leq i < i_{n+1}\}$ and $\tau_n\big|_{W_i \cup X_i}$ is connected for each i such that $i_n \leq i < i_{n+1}$.

Define $\nu(x,y) = \lim \mu_n(x,y)$. This map is a well defined metric since for any $x, y \in X$, $\nu(x,y) = \mu_m(x,y)$ for all

$$m \geq \max\{n : i_n \leq i < i_{n+1} \text{ and } x \in W_i \text{ or } y \in W_i\} \cup \{0\}.$$

Let σ be the topology generated by ν. To show that $\sigma \subset \tau$ we show that ν preserves convergent μ sequences. Suppose that $\{y_j : j \in \omega\}$ and y are such that $\lim_{j\to\infty} \mu(y, y_j) = 0$. If $y \notin K$ there are $n, m \in \omega$ such that $y, y_j \in X \setminus \mathrm{cl}(U_{k_n+1})$ for all $j \geq m$. Then, $\nu(y, y_j) = \mu_n(y, y_j)$ for all $j \geq m$, hence $\lim_{j\to\infty} \nu(y, y_j) = \lim_{j\to\infty} \mu_n(y, y_j) = 0$, since μ_n preserves μ convergent sequences. Now suppose that $y \in K$. If $y_j \in U_m$ for all m, let $m_j = j$. Otherwise, let $m_j = \max\{m : y_j \in U_m\}$. Since $\mu(y, y_j) \to 0$, $m_j \to \infty$. If there is $i \in \omega$ such that $y_j \in W_i$ then there is n such that $i \geq i_n \geq n = m_j$, since $y_j \in W_i \subset U_{m_j}$. In this case, by the consequence of (i) and (iii), $\nu(y, y_j) \leq \mu(y, y_j) + 6/2^n = \mu(y, y_j) + 6/2^{m_j}$. If $y_j \notin W_i$ for all $i \in \omega$ then $\nu(y, y_j) = \mu(y, y_j)$. In either case $\nu(y, y_j) \leq \mu(y, y_j) + 6/2^{m_j}$ and therefore $\nu(y, y_j) \to 0$. Note that $\nu\Big|_{(W_i \cup X_i)^2} = \mu_n\Big|_{(W_i \cup X_i)^2}$ for $i_n \leq i < i_{n+1}$. Hence $W_i \cup X_i$ is connected in σ. Furthermore, $X_i \cap W_j \neq \emptyset$ for each $j \in J_i$ and hence $W_i \cup \bigcup_{j \in X_i} W_j$ is connected as well. This means that W_i is 'linked' to W_0 for every i such that $1 = i_1 \leq i < i_2$ and since $\bigcup_{i\in\omega} J_i = \omega \setminus \{0\}$, any later W_i is 'linked' to W_0. Therefore any σ-clopen subset, Z, of X would have to be empty, or contain W_i for all i. Since $\bigcup_{i\in\omega} W_i$ is dense in X, Z is trivial. Hence σ is connected. $\square$

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